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Model parameters determination for the Gaia Basic Angle Monitoring device

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Contents

1 Abstract

The BAM (Basic Angle Monitoring) device is an interferometer dedicated to the measurement of the basic angle (hereafter, BA), i.e. the angle between the lines of sight of the two GAIA telescopes. Laser beams are projected towards each mirror; the beams are then splitted and forced to interfere separately onto two separate portions of a CCD. The BA is function of the differential of the maximum (ZOPD, zero Optical Path Difference) of the two interferogram w.r.t. their nominal position.
In this technical note, we first recall the interferometric model on which the analysis relies. This model depends on several parameters, both instrumental and observational. We then describe the algorithms for the estimation of the model parameters and of the ZOPD for an interferometric signal. We choose among different estimators for the performance of the fitting algorithms, and we finally test them with simulated interferometric images, under different noise conditions.

2 Introduction

The BAM Calibration set of algorithms is aimed at the estimation of the value of the most important parameters needed for the modeling of the image shape. The monochromatic bi-dimensional image model for each aperture is given by:

$$I(x',y') = A(D,f,\lambda,x',y') \cdot \left\{1 + V cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right]\right\} + bg \qquad (1)$$

where A is the envelope shape, D is the aperture diameter ([mm]), f is the focal length ([mm]), λ is the working wavelength ([mm]), x' e y' are the image coordinates ([mm]), V is the visibility (being function of the wavelength, we consider it as a constant), ϕ is the fringe phase ([rad]), inside the envelope, bg is the background noise. The image coordinates contain the displacement of the image w.r.t. the nominal position (x_0, y_0):

$$x' = x - x_0 + \Delta x_{BAV}$$

$$y' = y - y_0$$

where Δx_{BAV} is linked to the variation of the Basic Angle (BA) between the two telescopes, as seen by the aperture.
The algorithms set, described in section 5, gives an estimate of the following parameters:

- the envelope parameters, namely the intensity A, the amplitude σ, the phase ϕ and the background level bg
- the Visibility
- the fringe period

3 Data model and nominal configuration

We now give a description of the model used for data simulation and of the nominal configuration of the involved parameters.

The envelope shape of Eq. 1 can be modeled in different ways; we initially choose a circular gaussian envelope:

$$A = a \exp\left(-\frac{r'^2}{2\sigma^2}\right) \tag{2}$$

where $r' = \sqrt{x'^2 + y'^2}$ moves in the circle of radius σ.
This envelope is centered in the point $\mu' = (x'_0, y_0)$, with $x' = x_0 - \Delta x_{BAV}$.

$$I(x, y) = a \exp\left(-\frac{r'^2}{2\sigma^2}\right) \cdot \left\{1 + V cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right]\right\} + bg \tag{3}$$

The actual size of the image is 60 samples (along direction) per 360 samples (across direction). The units are not the same along the two directions. The pixel size is 10 μm (across) and 30 μm (along). The across unit corresponds to 1 pixel (10 μm), while the along unit corresponds to 2 pixels (60 μm). The resulting image is squared, but the samples are not equally distributed. This means, in particular, that the samples on the along direction, say y, are binned on chip.

4 Quality estimators

Since all tests are repeated over a number of images, say N, the results are summarized thanks to several estimators, described in this section. For every parameter x to be estimated, we call $\hat{x}_i$ the estimation over the i-*th* image of the value μ_i of the parameter. We will analyse the case 1) where $\mu_i = \mu$ does not vary over the images and 2) μ_i follows an assigned distribution with expectation $E(\mu_i) = \mu$ and variance $Var(\mu_i) = \sigma_\mu^2$. Moreover, we call $\hat{s}_i$ the estimation over the i-*th* image of the standard deviation $\sigma_{\hat{x}}$ of $\hat{x}$.

With simulated data, μ_i is in general known, while $\sigma_{\hat{x}}$ is not. With real data both μ_i and $\sigma_{\hat{x}}$ are not known. Therefore we will define either estimators that are function of μ_i and $\sigma_{\hat{x}}$ or estimators that rather use $\hat{s}_i$ and an estimation of μ_i instead. We give here a list of the estimators we consider useful for our analysis:

- Sample Mean (SM):
$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N}\hat{x}_i$$

- Sample Standard Deviation (SSD):
$$\bar{s}_s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\hat{x}_i - \bar{x})^2}$$

- Average Error (AVE):
$$\bar{\epsilon} = \frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i)$$

- Standard Deviation (STD):
$$\bar{s} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i)^2}$$

- true Normalized Average Error (tNAE):
$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\sigma_{\hat{x}}}$$

- true Normalized Standard Deviation (tNSD):
$$\bar{\varsigma}_t = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{(\hat{x}_i - \mu_i)^2}{\sigma_{\hat{x}}^2}}$$

Now, given that $\sigma_{\hat{x}}$ is usually not known, tNAE and tNSD cannot be computed. We will rather use the estimated value $\hat{s}_i$ and define four more estimators:

- Normalized Average Error (NAE):

$$\bar{\varepsilon} = \frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\hat{s}_i}$$

- Normalized Standard Deviation (NSD):

$$\bar{\varsigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\frac{(\hat{x}_i - \mu_i)^2}{\hat{s}_i^2}}$$

- sample Normalized Average Error (sNAE):

$$\bar{\varepsilon}_s = \frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i - \bar{x}}{\hat{s}_i}$$

- sample Normalized Standard Deviation (sNSD):

$$\bar{\varsigma}_s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\frac{(\hat{x}_i - \bar{x})^2}{\hat{s}_i^2}}$$

We list in the following the properties of each estimator for a general parameter x.

4.1 Properties of the quality estimators

First of all, we consider the estimation $\hat{x}_i$ of the parameter x from the i-*th* image. We make the basic assumption for $\hat{x}_i$ of being independent, identically distributed, and with finite expectation and variance. Moreover, we consider a possible bias caused by the estimation algorithm:

$$x_i = \mu_i + \hat{b}_i$$

where $\hat{b}_i$ is the estimate of the bias associated to the i-*th* image. Assuming that the bias is such that it follows a statistical distribution with expectation $E[\hat{b}_i] = b$ and variance $\mathrm{var}(\hat{b}_i) = \sigma_b^2$, we have:

$$\begin{aligned} E[x_i] &= E[\mu_i] + E[\hat{b}_i] = \mu + b \\ \sigma_{\hat{x}}^2 &= E[(\hat{x}_i - E[\hat{x}_i])^2] = E[(\mu_i + \hat{b}_i - \mu - b)^2] = \\ &= E[(\mu_i - \mu)^2 + (\hat{b}_i - b)^2 + 2(\mu_i - \mu)(\hat{b}_i - b)] = \sigma_\mu^2 + \sigma_b^2 + 2\sigma_{\mu\, b} \end{aligned} \tag{4}$$

As we can expect, the variance of the estimation is a combination of the statistical moments of the parameter's distribution and of the estimating features. We now analyze in details the properties of most of the estimators define in the previous section.

- Sample Mean:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} \hat{x}_i = \frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i + \mu_i) = \frac{1}{N}\sum_{i=1}^{N}(\hat{b}_i + \mu_i)$$

The expectation of the Sample Mean is given by:

$$\mu_{\bar{x}} \equiv E(\bar{x}) = \frac{1}{N}\sum_{i=1}^{N}[E(\hat{b}_i) + E(\mu_i)] = \frac{1}{N}\sum_{i=1}^{N}(b + \mu) = b + \mu \tag{5}$$

Thanks to our assumption on the distribution of the estimates $\hat{x}_i$, then the Sample Mean is asymptotically normally distributed with $E(\bar{x}) = \mu_{\bar{x}}$ and variance $\sigma_{\bar{x}}^2$ given by

$$\sigma_{\bar{x}}^2 = \frac{\sigma_{\hat{x}}^2}{N} \tag{6}$$

This information could be used for the estimation of a confidence interval for $\mu_{\bar{x}}$, with confidence level $1 - \alpha$. Since in practice $\sigma_{\hat{x}}^2$ is not known, and we only have its estimates $\hat{s}_s^2$, we consider the following confidence interval:

$$\mu_{\bar{x}} \in \left(\bar{x} - t_{\alpha/2,N-1}\frac{\bar{s}_s}{\sqrt{N}}, \bar{x} + t_{\alpha/2,N-1}\frac{\bar{s}_s}{\sqrt{N}}\right) \tag{7}$$

where $t_{\alpha/2,N-1}$ is the quantile of the Student t distribution with $\alpha/2, N-1$ parameters.

- Average error:

$$\bar{\epsilon} = \frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i)$$

The expectation $\mu_{\bar{\epsilon}}$ of the Average Error is:

$$E(\bar{\epsilon}) \equiv \mu_{\bar{\epsilon}} = \frac{1}{N}\sum_{i=1}^{N} E(\hat{b}_i) = b$$

as the contribution of the true values is cancelled out. The variance $\sigma_{\bar{\epsilon}}^2$ is given by:

$$\sigma_{\bar{\epsilon}}^2 = \mathrm{var}\left(\frac{1}{N}\sum_{i=1}^{N}(\hat{b}_i)\right) = \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{var}(\hat{b}_i) = \frac{\sigma_b^2}{N}$$

thanks to the assumption of independence for the estimates $\hat{x}_i$. Moreover, since we assume that $\hat{x}_i$ are also equally distributed, the Average Error is asymptotically normally distributed with $E(\bar{\epsilon}) = \mu_{\bar{\epsilon}}$ and variance $\sigma_{\bar{\epsilon}}^2$.
From eq. 8 we obtain that $\sigma_b^2 = N\sigma_{\bar{\epsilon}}^2$.
If the bias contribution can be neglected, than $\mu_{\bar{\epsilon}} \sim 0$.
The Sample Mean and the Average Error provide basically the same information. However the Average Error can be used only with simulated data, where μ_i is known. We give also a different formulation for $\sigma_{\bar{\epsilon}}^2$:

$$\begin{aligned} \sigma_{\bar{\epsilon}}^2 &= \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{var}(\hat{x}_i - \mu_i) = \frac{1}{N^2}\sum_{i=1}^{N}[\mathrm{var}(\hat{x}_i) + \mathrm{var}(\mu_i) - 2\sigma_{\hat{x}\,\mu}] = \\ &= \frac{\sigma_{\hat{x}}^2 + \sigma_{\mu}^2 - 2\sigma_{\hat{x}\,\mu}}{N} \end{aligned} \tag{8}$$

- Standard Deviation:

$$\bar{s} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i)^2}$$

Using the squared Standard Deviation and recalling previous definitions we can write:

$$\bar{s}^2 = \frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i)^2 = \frac{1}{N}\sum_{i=1}^{N}\hat{b}_i^2$$

The expected value of the squared Standard Deviation is then

$$E(\bar{s}^2) = \frac{1}{N}\sum_{i=1}^{N} E(\hat{b}_i^2) = \frac{1}{N}\sum_{i=1}^{N}[\sigma_b^2 + E^2(\hat{b}_i)] = \sigma_b^2 + b^2$$

or also, taking into account that $E(\hat{x}_i) = \mu_i + b_i$,

$$E(\bar{s}^2) = E\left[\frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu_i)^2\right] = \frac{1}{N}\sum_{i=1}^{N} E[(\hat{x}_i - \mu_i)^2] =$$

$$= \frac{1}{N}\sum_{i=1}^{N} E[(\hat{x}_i - \mu - b - \mu_i + \mu + b)^2] = \frac{1}{N}\sum_{i=1}^{N} E[\{(\hat{x}_i - \mu - b) - (\mu_i - \mu) + b\}^2] =$$

$$= \frac{1}{N}\sum_{i=1}^{N}\{E[(\hat{x}_i - \mu - b)^2] + E[(\mu_i - \mu)^2] + E[b^2] - 2E[(\hat{x}_i - \mu - b)(\mu_i - \mu)] +$$

$$+ 2E[b(\hat{x}_i - \mu - b)] - 2E[b(\mu_i - \mu)]\} =$$

$$= \frac{1}{N}\sum_{i=1}^{N}\{\sigma_{\hat{x}}^2 + \sigma_\mu^2 + b^2 - 2\sigma_{\hat{x}\,\mu} + 2bE[(\hat{x}_i - \mu - b)] - 2bE[(\mu_i - \mu)]\}$$

where the two last factors are null since the expectations go to zero. We finally obtain:

$$E(\bar{s}^2) = \sigma_{\hat{x}}^2 + \sigma_\mu^2 + b^2 - 2\sigma_{\hat{x}\,\mu} = b^2 + N\sigma_{\bar{\epsilon}}^2$$

having used Eq. 8.

We can define a confidence interval for $\bar{s}^2$, considering that, under the condition that $\hat{x}$ is normally distributed, with mean $\mu_{\hat{x}}$ and variance $\sigma_{\hat{x}}^2$, then $N\frac{\bar{s}^2}{\sigma_{\hat{x}}^2} \sim \chi_N^2$, where χ_N^2 has a Chi-square distribution with parameter N. For N sufficiently large, the χ_N^2 distribution can be approximated by a gaussian, with mean N and variance $2N$.
In practice, since we do not know $\sigma_{\hat{x}}$, we shall use the estimates $\hat{s}_i^2$ instead, assuming that we can neglect the contribution of the bias.

- Sample Standard Deviation:

$$\bar{s}_s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\hat{x}_i - \bar{x})^2}$$

When the true values μ_i are not known, the Sample Standard Deviation can be used instead of the Standard Deviation. Remembering our assumptions on $\hat{x}_i$, its expectation is given by:

$$\begin{aligned} E[\bar{s}_s^2] &= \frac{1}{N-1}\sum_{i=1}^{N} E[(\hat{x}_i - \bar{x})^2] = \quad (9) \\ &= \frac{1}{N-1}\sum_{i=1}^{N} E[(\hat{x}_i - \bar{x} + \mu + b - \mu - b)^2] = \\ &= \frac{1}{N-1}\sum_{i=1}^{N} E[\{(\hat{x}_i - E[\hat{x}_i]) - (\bar{x} - E[\bar{x}])\}^2] = \\ &= \frac{1}{N-1}\sum_{i=1}^{N} \{\sigma_{\hat{x}}^2 + \sigma_{\bar{x}}^2 - 2E[(\hat{x}_i - E[\hat{x}_i])(\bar{x} - E[\bar{x}])]\} \end{aligned}$$

To evaluate the last factor, we remember that $\bar{x} = \frac{\sum_{j=1}^{N}\hat{x}_j}{N}$, and $E[\bar{x}] = \frac{\sum_{j=1}^{N}E[\hat{x}_j]}{N}$, and we obtain:

$$\begin{aligned} E[(\hat{x}_i &- E[\hat{x}_i])(\bar{x} - E[\bar{x}])] = E[(\hat{x}_i - E[\hat{x}_i])(\frac{1}{N}\sum_{j=1}^{N}(\hat{x}_j - E[\hat{x}_j]))] = \\ &= E[\frac{1}{N}\sum_{j=1}^{N}(\hat{x}_i - E[\hat{x}_i])(\hat{x}_j - E[\hat{x}_j])] = \\ &= \frac{1}{N}\{E[(\hat{x}_i - E[\hat{x}_i])^2] + 2\sum_{j=1, j\neq i}^{N}\mathrm{cov}(\hat{x}_i, \hat{x}_j)\} = \frac{\sigma_{\hat{x}}^2}{N} \quad (10) \end{aligned}$$

since we have assumed the independence of estimates. Substituting in Eq. 9, remembering that $\sigma_{\bar{x}}^2 = \frac{\sigma_{\hat{x}}^2}{N}$, we obtain:

$$E[\bar{s}_s^2] = \frac{1}{N-1}\sum_{i=1}^{N}\{\sigma_{\hat{x}}^2 + \frac{\sigma_{\hat{x}}^2}{N} - 2\frac{\sigma_{\hat{x}}^2}{N}\} = \sigma_{\hat{x}}^2 \quad (11)$$

that is, $\bar{s}_s$ is a correct estimator of $\sigma_{\hat{x}}^2$.

- true Normalized Average Error:

$$\bar{\varepsilon}_t = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\sigma_{\hat{x}}} = \frac{1}{N}\sum_{i=1}^{N}\hat{x}_i^*$$

Assuming zero bias, $\hat{x}_i^*$ is a standardized random variable, and so we expect the tNAE to have zero mean.
In case of non-zero bias b_i for each image, given that in this case the variance $\sigma_{\hat{x}}$ is supposed known, the expected value of the tNAE is given by:

$$\mu_{\bar{\varepsilon}_t} = E\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\sigma_{\hat{x}}}\right) = \frac{1}{N}\sum_{i=1}^{N}\frac{E[\hat{b}_i]}{\sigma_{\hat{x}}} = \frac{b}{\sigma_{\hat{x}}}$$

Under the same assumptions, the variance of the tNAE is:

$$\sigma^2_{\bar{\varepsilon}_t} = \operatorname{var}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\sigma_{\hat{x}}}\right) = \frac{1}{N}\sum_{i=1}^{N}\frac{\operatorname{var}(\hat{x}_i - \mu_i)}{\sigma_{\hat{x}}} = \frac{N\sigma_b^2}{N^2\sigma_{\hat{x}}} =$$

$$= \frac{\sigma_b^2}{N\sigma_{\hat{x}}} = \frac{1}{N}\frac{\sigma_b^2}{\sigma_\mu^2 + \sigma_b^2 + 2\sigma_{\mu\, b}}$$

If $N \to \infty$, then $\operatorname{var}(\bar{\varepsilon}_t)$ goes to zero.
For the assumption on $\hat{x}_i$, the tNAE has an asymptotic normal distribution with mean and variance estimated as above. We can then construct a confidence interval, providing the same information of the Average Error. However, since $\sigma^2_{\hat{x}}$ is not known, the Normalized Average Error will be used in practice instead.

- true Normalized Standard Deviation:

$$\bar{\varsigma}_t = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{(\hat{x}_i - \mu_i)^2}{\sigma^2_{\hat{x}}}}$$

As for the Standard Deviation, assuming that the bias can be considered negligible, the quantity $N \cdot tNSD^2$ has a χ^2_N distribution, and for large N we can approximate it with a normal distribution with mean N and variance $2N$. We can use this information to construct a confidence interval. Again, when $\sigma^2_{\hat{x}}$ is not known, the Normalized Standard Deviation can be used instead. The information should be equivalent to the one provided by the Standard Deviation.

- Normalized Average Error:

$$\bar{\varepsilon} = \frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\hat{s}_i}$$

Since $\bar{\varepsilon}$ is the ratio of two random variables, its properties can be derived at least at first order, but this is out of the scope of this work. So we use a stronger assumption, approximating $\hat{s}_i$ with $\sigma_{\hat{x}}$. The properties are therefore the same as $\bar{\varepsilon}_t$.

- Normalized Standard Deviation:

$$\bar{\varsigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\frac{(\hat{x}_i-\mu_i)^2}{\hat{s}_i^2}}$$

As for $\bar{\varepsilon}$, we approximate $\hat{s}_i$ with $\sigma_{\hat{x}}$.

- sample Normalized Average Error:

$$\bar{\varepsilon}_s = \frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i-\bar{x}}{\hat{s}_i}$$

If we assume $\hat{s}_i = \sigma_{\hat{x}}$ to be known, we can derive the properties of the following estimator:

$$\bar{\varepsilon}_s^* = \frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i-\bar{x}}{\sigma_{\hat{s}}}$$

that we can call sNAE*.

$$\mu_{\bar{\varepsilon}_s^*} = E\left(\frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i-\bar{x}}{\sigma_{\hat{x}}}\right) = \frac{1}{N}\frac{1}{\sigma_{\hat{x}}}\sum_{i=1}^{N}E[(\hat{x}_i-\bar{x})] = 0$$

since $\hat{x}_i = \bar{x}$.
The variance of the sNAE* is:

$$\begin{aligned}\sigma^2_{\bar{\varepsilon}_s^*} &= \text{var}\left(\frac{1}{N-1}\sum_{i=1}^{N}\frac{\hat{x}_i-\bar{x}}{\sigma_{\hat{x}}}\right) = \frac{1}{(N-1)^2}\frac{1}{\sigma^2_{\hat{x}}}\sum_{i=1}^{N}\text{var}(\hat{x}_i-\bar{x}) = \\ &= \frac{1}{(N-1)^2}\frac{1}{\sigma^2_{\hat{x}}}\sum_{i=1}^{N}E[(\hat{x}_i-\bar{x})^2]\end{aligned}$$

From Eq. 11 we can write $E[(\hat{x}_i-\bar{x})^2] = \frac{N-1}{N}\sigma^2_{\hat{x}}$, and substituting we obtain:

$$\sigma^2_{\bar{\varepsilon}_s^*} = \frac{1}{(N-1)^2}\frac{1}{\sigma^2_{\hat{x}}}\sum_{i=1}^{N}\frac{N-1}{N}\sigma_{\hat{x}} = \frac{1}{(N-1)^2}\frac{1}{\sigma^2_{\hat{x}}}N\frac{N-1}{N}\sigma^2_{\hat{x}} = \frac{1}{N-1}$$

which tends to zero as $N \to \infty$. Thanks to the assumption on $\hat{x}_i$, sNAE* is asymptotically normally distributed with mean $\mu_{\bar{\varepsilon}_s^*}$ and variance $\sigma^2_{\bar{\varepsilon}_s^*}$. This is the only information we have for evaluating a confidence interval for sNAE.

- sample Normalized Standard Deviation:

$$\bar{\varsigma}_s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\frac{(\hat{x}_i - \bar{x})^2}{\hat{s}_i^2}}$$

As before, we can assume $\hat{s}_i = \sigma_{\hat{x}}$ to be known, and derive the properties of the following estimator:

$$\bar{\varsigma}_s^* = \frac{1}{N-1}\sum_{i=1}^{N}\frac{(\hat{x}_i - \bar{x})^2}{\sigma_{\hat{s}}^2}$$

that we can call sNSD*.
The mean of this estimator is:

$$\begin{aligned} E(\bar{\varsigma}_s^*) &= \frac{1}{N-1}\sum_{i=1}^{N}\frac{E[(\hat{x}_i - \bar{x})^2]}{\sigma_{\hat{s}}^2} = \frac{1}{N-1}\frac{1}{\sigma_{\hat{s}}^2}\sum_{i=1}^{N}E[(\hat{x}_i - \bar{x})^2] = \\ &= \frac{1}{N-1}\frac{1}{\sigma_{\hat{s}}^2}\sum_{i=1}^{N}\frac{N-1}{N}\sigma_{\hat{x}}^2 = 1 \end{aligned}$$

The variance involves the fourth moments.

4.2 Application to specific cases

4.2.1 Noise free images with fixed parameter

In this case $\mu_i = \mu, \quad \forall i$. Since there is no variability, we expect to be dominated by the bias effects.
The parameter are fixed and the images are noiseless, so the estimates $\hat{x}_i$ are equal for all i. Their definitions simplify: $\hat{x}_i = \mu + \hat{b}$, since the bias should be constant for the lack of variability over each image (a part from numerical approximations). So the moments of the distribution of $\hat{x}_i$ become:

$$\begin{aligned} E[\hat{x}_i] &= \mu + b \\ \sigma_{\hat{x}}^2 &= \sigma_\mu^2 + \sigma_b^2 + 2\sigma_{\mu\,b} = 0 \end{aligned}$$

The estimators description is simplified as follows:

- Sample Mean:

$$\bar{x} = \mu + \hat{b}$$

Since the estimates $\hat{x}_i$ are equal for all i, also $\hat{b}$ is constant for all images, and we obtain:

$$E(\bar{x}) = \mu + E(\hat{b}) = \mu + b$$

where $b = E[\hat{b}] = \hat{b}$. The variance $\sigma^2_{\bar{x}}$ is of course 0.

- Average error:

$$\bar{\epsilon} = \bar{x} - \mu = \hat{b}$$

Since $\bar{\epsilon}$ is the same for each image:

$$E(\bar{\epsilon}) = E(\hat{b}) = b \tag{12}$$

Therefore the Average Error provides an estimate of the bias.

- Standard Deviation:

$$\bar{s} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{x}_i - \mu)^2}$$

In this particular case, $\bar{s} = |\bar{\epsilon}|$, so Standard Deviation does not add information.

- Sample standard deviation:

$$\bar{s}_s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\hat{x}_i - \bar{x}_N)^2} = 0$$

Since the N images are noiseless, all $\hat{x}_i = \hat{x}$ are equal to $\bar{x}$, so $\bar{s}_s$ is zero. The bias is canceled out by the subtraction.

- Normalized Average Error:

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x} - \mu}{\hat{s}_i}$$

Again in this case we have $\hat{x}_i = \hat{x}$, $\mu_i = \mu$, $\forall i$. We can also assume that the only contribution to the error associated to the parameter estimation comes from the estimation process itself. Under this hypothesis, $\hat{s}_i = \hat{s}$, $\forall i$:

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x}-\mu}{\hat{s}} \sim \frac{\hat{b}}{\hat{s}}$$

and we obtain an indication of the relative weight of the bias b with respect to the error associated to the parameter estimation algorithm $\sigma_{\hat{x}}$.

- Normalized Standard Deviation:

$$\bar{\varsigma} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{(\hat{x}-\mu)^2}{\hat{s}^2}} \sim \frac{|\hat{b}|}{|\hat{s}|} = |\bar{\varepsilon}|$$

  As for the Standard Deviation, the NSD does not add further information.

- sNAE and sNSD: they are identically zero because the bias contribution is cancelled out

4.2.2 Noise free images with varying parameter

The parameter is allowed to vary around a certain value with a known variance, following a known distribution (true parameter values: μ_i, with $E(\mu_i) = \mu$, $var(\mu_i) = \sigma_\mu^2$, for $i = 1, \ldots, N$). Given that the images are noiseless, we expect to be dominated by the statistics of the estimator rather than by the statistic of the parameter. This means that we expect $\hat{s}_i$ to vary around a mean value which value depends upon the distribution of the parameter, with a distribution which is not easily derived as it is dominated by the parameter estimator properties. And we expect to be again dominated by the bias effect.

- Sample Mean:

$$E(\bar{x}) = \frac{1}{N}\sum_{i=1}^{N}E(\mu_i) + \frac{1}{N}\sum_{i=1}^{N}E(\hat{b}_i) = \mu + \frac{1}{N}\sum_{i=1}^{N}b_i$$

- Average error:

$$\bar{\epsilon} = \frac{1}{N}\sum_{i=1}^{N}b_i$$

  It can be considered as an estimate of the mean bias, assumed different for different values of the parameter to be estimated.

- Standard Deviation:

$$\bar{s} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\hat{b}_i^2}$$

Since we are dominated by the bias, the Standard Deviation is not a good estimation of $\sigma_{\hat{x}}$.

- Normalized Average Error:

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{x}_i - \mu_i}{\hat{s}_i}$$

The situation is similar to the case of fixed parameters. We have now

$$\bar{\varepsilon} = \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{b}_i}{\hat{s}_i}$$

that gives us an indication of the mean relative weight of the bias $\hat{b}_i$ with respect to the error associated to the parameter estimation algorithm $\hat{s}_i$.

- Normalized Standard Deviation:

$$\bar{\varsigma} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{(\hat{x}_i - \mu_i)^2}{\hat{s}_i^2}}$$

The situation is again similar to the case of fixed parameters. We have here

$$\bar{\varsigma} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{\hat{b}_i^2}{\hat{s}_i^2}}$$

in this case however NSD is no longer simply the absolute value of NAE.

- sNAE and sNSD: bias do not cancel out anymore.

4.2.3 Noisy images

We are here in presence of a noisy process, which is simulated as usual, i.e. a poisson distribution with $\sigma_{ph}^2 = \mu_{ph}$ for each image sample. However, the dependence of $\sigma_{\hat{x}}$ vs. the poisson variance σ_{ph}^2 is not easily derived. Again $\hat{s}_i$ varies over the images, the distribution being this time dominated by the poisson noise. Therefore in this case the definition of confidence intervals are useful to verify: 1) that we can neglect the bias introduced by the estimation process and 2) that our hypothesis of asymptotic normal distributions, whenever made, cannot be rejected by the results. We will assume and verify if NAE follows a normal distribution with $(\mu, \sigma^2) = (0, N^{-1})$ and NSD^2 follows a normal distribution with $(\mu, \sigma^2) = (1, 2 \cdot N^{-1})$ at a given level of confidence.

4.3 Criteria

In this subsection, we use the discussion made in 4.2 to derive general criteria for evaluation of the algorithm performances. They are summarized in Table 1.
Definitions:

- Q_b is the maximum allowed relative bias level. For the purpose of this document, we set initially $Q_b = 1.0 \cdot 10^{-4}$
- n_σ is the level of confidence required. In our simulations we require $n_\sigma = 3$ which means 3-σ level of confidence.
- N is the number of images used for a given test. In our case, $N = 1000$ for all test involving noise.

Noise free images - fixed parameters
$AVE = \hat{b} < Q_b \cdot \mu$ if $\mu \neq 0$
$AVE = \hat{b} < Q_b$ if $\mu = 0$
Noisy images
$NAE \sim Gauss\left(0, \frac{1}{N}\right) \implies \lvert NAE \rvert < n_\sigma \cdot \sqrt{\frac{1}{N}} \sim 0.095$
$NSD^2 \sim \frac{\chi^2}{N} \sim Gauss\left(1, \frac{2}{N}\right) \implies \lvert NSD^2 - 1 \rvert < n_\sigma \cdot \sqrt{\frac{2}{N}} \sim 0.134$

TABLE 1: Criteria

5 Algorithm description

In this section we describe the algorithms used for the determination of parameters.

5.1 Envelope shape

Given the 2D image, if we integrate it in the along scan direction, we can have direct access to the envelope shape. From Eq. (1), integrating along the x variable, we obtain:

$$\begin{aligned} S(y') &= \int_x I(x',y')dx' = \\ &= \int_{x_1}^{x_2} (a \exp\left(-\frac{x'^2+y'^2}{2\sigma^2}\right) \cdot \left\{1 + V cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right]\right\} + bg) dx' = \\ &= a \exp\left(-\frac{y'^2}{2\sigma^2}\right) \int_{x_1}^{x_2} \exp\left(-\frac{x'^2}{2\sigma^2}\right) dx' + \\ &+ aV \exp\left(-\frac{y'^2}{2\sigma^2}\right) \int_{x_1}^{x_2} \exp\left(-\frac{x'^2}{2\sigma^2}\right) \cdot cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right] dx' + \\ &+ \int_{x_1}^{x_2} bg\ dx' = a \exp\left(-\frac{y'^2}{2\sigma^2}\right)(H + VK) + \Delta x'\ bg \end{aligned} \tag{13}$$

with $H = \int_{x_1}^{x_2} \exp\left(-\frac{x'^2}{2\sigma^2}\right) dx'$, $K = \int_{x_1}^{x_2} \exp\left(-\frac{x'^2}{2\sigma^2}\right) \cdot cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right] dx'$, and $\Delta x' = x_2 - x_1$.

Using the last equality of Eq. (13), we can model the envelope shape by means of an exponential function with four parameters (A, μ, S and D):

$$F(y) = A \exp\left(-\frac{(y-\mu)^2}{2S^2}\right) + D \tag{14}$$

where A and D are linked to a and bg thanks to the following relations:

$$a = A/(H + VK) \tag{15}$$

$$bg = D/\Delta x' \tag{16}$$

while μ and S give respectively the *mean* displacement of the envelope along the y coordinate and the width of the envelope.

If the integration interval contains a complete number of fringes, the modulated integral K vanishes, and Eq. 15 reduces to $a = A/H$. If this is not the case, K introduces a perturbation strictly related to the sampling of the fringes. This may lead in problems in determining the value of the envelope original amplitude.

In our simulations, the integration interval $[x_1, x_2]$ is the samples space with 60 points. The integration in the along scan direction is done summing up the 360 interferograms.

The estimation of the parameters is done by two means:

- a *Maximum Likelihood* (ML) approach we implemented in IDL for the purpose of this document;
- the IDL *Curvefit* function, which performs a non-linear least squares fit, to provide a cross-check of the results using IDL built-in functions. The description of the *Curvefit* algorithm can be found at the following link: *http://www.ittvis.com/portals/0/pdfs/idl/refguide.pdf*
  Curvefit requires at input the setting of weights; we choose unitary weights.

5.2 Fringe shape

As in the previous section, we start again from the 2D image, but now we integrate in the across scan direction. In this way, we obtain an integrated interferogram. From Eq. 1, integrating along the y variable, we obtain:

$$
\begin{aligned}
F(x') &= \int_y I(x', y') dy' = \\
&= a \exp\left(-\frac{x'^2}{2\sigma_x^2}\right) \int_{y_1}^{y_2} \exp\left(-\frac{y'^2}{2\sigma_y^2}\right) dy' + \\
&+ aV \exp\left(-\frac{x'^2}{2\sigma_x^2}\right) cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right] \int_{y_1}^{y_2} \exp\left(-\frac{y'^2}{2\sigma_y^2}\right) dy' + \\
&+ \int_{y_1}^{y_2} bg\ dy' = A^* \exp\left(-\frac{x'^2}{2\sigma_x^2}\right) \left\{1 + V cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right]\right\} + B^*
\end{aligned}
$$

where $A^* = a \int_{y_1}^{y_2} \exp\left(-\frac{y'^2}{2\sigma^2}\right) dy'$ and $B^* = \Delta y \cdot bg$. Since $x' = x - x'_0$, we can write the last equality as:

$$F(x) = A^* \exp\left(-\frac{(x-x_0')^2}{2\sigma_x^2}\right)\left\{1 + V cos\left[2\pi\frac{B}{\lambda\cdot f}(x-x_0')+\phi\right]\right\} + B^* \quad (17)$$

Starting from this equation, we would like to estimate the following parameters:

- A^*: the integrated intensity
- x_0': the difference between the actual and the nominal zero OPD position
- σ_x^2: the amplitude of the envelope in the across scan direction
- V: the fringe visibility
- ϕ: the phase shift between the envelope peak and the white fringe

In our simulations, the integration interval $[y_1, y_2]$ is the 360-pixels space. The integration on the across direction is done summing along the samples direction.
As in previous section, parameter estimation is done with both *Curvefit* and *Maximum Likelihood.*

5.3 Fringe period

We estimate the period of the fringe by Fourier transforming the signal in the Along scan direction where the information is stored. This can be done both on the 2D fringe pattern (Eq. 1) and on the same signal integrated in the Across scan direction (Eq. 17). In the first case we obtain N_r independent estimation of the fringe period, where N_r is the number of AC samples. In the latter case, we have one single estimation, which takes directly advantage from the improved signal to noise ratio.
Another available option is to use a densified image. The algorithm we developed can be set to add zero value samples at the AL borders of the window; the number of added samples can be any integer number of the AL window size, up to a maximum value currently set at 9. We call this the *densification factor* d_f. The result of densifying the image by a factor d_f is a correspondent densification in the Fourier space, decreasing the sampling resolution by the same factor and therefore adding useful information in the Fourier transform function. The price we have to pay is of course an increased computational

load and a different energy distribution. An alternative approach would be to directly compute the discrete Fourier transform for an arbitrary number of value of the Fourier independent variable; this is currently not implemented in our simulations.

The steps for Fringe period estimation are the following:

- generate the densified image (only if $d_f > 1$)
- compute the Fast Fourier Transform on either the 1D or 2D fringe pattern;
- fit the FFT result to a suitable analytical model
- derive estimates of the fringe period and associated standard error from fit results.

The analytical model used to fit the Fourier transform in our simulations is currently a simple gaussian profile.

In our simulations we perform four different estimates of the fringe period, combining the 1D/2D option with the densified/non-densified image (with respectively $d_f = 1$ and $d_f = 9$).

5.4 Nominal and non-nominal values

We recall the 2-dimensional image formula:

$$I(x', y') = a \exp\left[-\left(\frac{x'^2}{2\sigma_x^2} + \frac{y'^2}{2\sigma_y^2}\right)\right] \cdot \left\{1 + V cos\left[2\pi \frac{B}{\lambda \cdot f} x' + \phi\right]\right\} + bg \quad (18)$$

with $x' = x - x'_0 = x - (x_0 + \Delta x_{BAV})$ and $y' = y - y'_0$. Table 2 lists the nominal value of all parameters involved, with the indication of the algorithm estimating them. Tables 3 and 4 list the values used when non-nominal configurations are used.

6 Tests

6.1 Tests description

In this section we describe the tests performed. The first are useful to check the limits of the algorithms from the computational point of view, giving an

par	nominal value	units	Algorithm
A	9e4	[el]	*not estimated*
σ_x	1.5	[mm]	Fringe Shape
σ_y	1.5	[mm]	Envelope Shape
x'_0	0	[mm]	Fringe Shape
y'_0	0	[mm]	Envelope Shape
V	1		Fringe Shape
ϕ	0	[rad]	Fringe Phase
bg	1e4	[el]	Envelope Shape
FP	0.0497	[mm]	Fringe Period
λ	852e-6	[mm]	*not estimated*
f	35e3	[mm]	*not estimated*
B	600	[mm]	*not estimated*

TABLE 2: Nominal values for image parameters.

estimate of the parameters variance due to numerical limits. We then assign a variability to one or more parameters, in order to identify the parameter variance due to its intrinsic perturbation. Finally, we let vary all parameters all together, to see the overall error associated to the estimate.

00: This test is focused on the envelope shape only. The model is a simplified 1-dimensional one:

$$I(x) = A \exp\left(-\frac{(x-\mu)^2}{2S^2}\right) + D$$

Parameters A, μ, D are at their nominal values, while S varies in the interval [6,35] samples, corresponding to [0.36,2.1] mm (nominal value, S=25 samples, that is, 1.5 mm). This test is aimed in particular to the estimation of the background D. If S is small, the envelope bell is smaller, and it is easier to estimate the background. Table 5 shows the nominal values of the envelope parameters for this particular test case. The algorithm is based on the IDL *curvefit* function.

0: this test should verify the perturbation on parameters' estimation introduced by a non integer number of fringes. It is based on the model of Eq. 18, and the fitting algorithms use the IDL *Curvefit* function.

0a: S varies in the interval [0.2,2.4] mm. The other parameters are set at their nominal values.

par	nominal value	units	Algorithm
A	9e4	[el]	*not estimated*
σ_x	1.500024438	[mm]	Fringe Shape
σ_y	1.500024438	[mm]	Envelope Shape
x'_0	0	[mm]	Fringe Shape
y'_0	1.44219985e-2	[mm]	Envelope Shape
V	0.81020838		Fringe Shape
ϕ	7.60197499e-3	[rad]	Fringe Phase
bg	1.000052148	[el]	Envelope Shape
FP	0.049700599	[mm]	Fringe Period
λ	852e-6	[mm]	*not estimated*
f	35e3	[mm]	*not estimated*
B	600	[mm]	*not estimated*

TABLE 3: non-nominal values (fixed) for image parameters.

0b: S varies in the interval [0.2,2.4] mm. The other parameters are not nominal, but fixed such that the fringe period contains an integer number of pixel, and the entire scan contains an integer number of fringes. In particular, λ = 800 nm, F = 50 m and B = 500 mm, instead of 850 nm, 35 m and 600 mm, respectively. The fringe period, defined as $f = \lambda F/B$, is equal to 80μm instead of 49.7μm. Since the pixel is 10μm wide, the fringe period spans 8 pixels; since interferograms span 360μm, each of them contains exactly 4 fringes.

01: generation of 100 images, with all parameters set at their nominal values; images are not noisy (RON, but not photon noise). This test is looking for variability caused only by computational problems, that should be avoided. It is based on model of Eq. 18, and the fitting algorithms use both the IDL *Curvefit* function and the *ML* approach.

1: generation of 1000 images, with all parameters set at their nominal values; images are noisy (photon noise and RON). This test is aimed at giving an estimate of the variability associated at each parameter, caused by noise on image. This is a lower limit under which intrinsec variability of parameters is not retrievable because of noise on data. It is based on model of Eq. 18, and the fitting algorithms use both the IDL *Curvefit* function and the *ML* approach.

par	nominal value	sigma	units	Algorithm
A	9e4	9	[el]	*not estimated*
σ_x	1.500024438	1.5e-5	[mm]	Fringe Shape
σ_y	1.500024438	1.5e-5	[mm]	Envelope Shape
x'_0	-2.92e-5	5e-3	[mm]	Fringe Shape
y'_0	1.44219985e-2	1.5e-2	[mm]	Envelope Shape
V	0.81020838	0.1		Fringe Shape
ϕ	7.60197499e-3	0.1	[rad]	Fringe Phase
bg	1.000052148	1	[el]	Envelope Shape
FP	0.049700599	-	[mm]	Fringe Period
λ	852e-6	1e-10	[mm]	*not estimated*
f	35e3	3.5	[mm]	*not estimated*
B	600	6e-2	[mm]	*not estimated*

Table 4: non-nominal values (gaussian distributed) for image parameters.

par	nominal value	units
A	9e4	[el]
μ	0	[samples]
S	25	[samples]
D	1e4	[el]

Table 5: Nominal values for envelope parameters.

2: generation of 1000 images with noise (photon noise and RON). Parameters have constant values, even if non nominal. The only parameter that is free to vary is alpha, i.e. the mirror tilt, from which it is possible to recovery the basic angle.
This test is aimed at giving an estimate of the variability associated at each parameter, caused not only by noise, but also from a non nominal value of the model variables. It is based on model of Eq. 18, and the fitting algorithms use both the IDL *Curvefit* function and the *ML* approach.

3: generation of 1000 images with noise (photon noise and RON). Parameters vary over the images, except for α, which has a constant but non nominal value (i.e., $\alpha \neq 0$). It is based on model of Eq. 18, and the fitting algorithms use both the IDL *Curvefit* function and the *ML* approach.

4: generation of 1000 images with noise (photon noise and RON). Parameters vary over the images, including α. It is based on model of Eq. 18, and the fitting algorithms use both the IDL *Curvefit* function and the *ML* approach.

6.2 Tests results

In this chapter we summarize the numerical results of all tests and provide comments for each one.

6.2.1 Test 00

Results are shown in table 6. The mean error is fairly around zero for μ and S, while there is some crosstalk between A and D.

par	mean error	std. dev.	aver. norm. err.	std. norm. err.
A	-0.0755208	0.298324	-13.1351	64.0534
μ	0.000000	0.000000	0.000000	0.000000
S	-6.18299e-6	6.13202e-5	-1.73280	18.2855
D	0.0771159	0.302339	16.2995	76.5564

TABLE 6: Estimation of envelope parameters (Test 00)

6.2.2 Test 0a

Results are shown in Table 7. Indipendently from the value of σ, the background value is estimated correctly, while none of the three available ways of retrieving the value A of the amplitude (i.e. application of a correction factor, numerical integration, analytical error function, shown in the last three columns) works.

6.2.3 Test 0b

Results are shown in Table 8. The numerical values are different from the previous case, but the result is basically the same. In this case, the value of A is overestimated rather than underestimated when σ increases. For this reason, in the next tests the estimation of the amplitude has not been further investigated. This is not an issue, as the amplitude has anyhow to be estimated in the process of determining the phase shift.

σ	bg	σ_{bg}	x'_0	$\sigma_{x'_0}$	corrFactA	Integral	Err function
0.2	9999.99	2.22e-3	0.0	2.24e-6	89347.2	89673.4	89673.2
0.3	9999.99	1.1e-3	0.0	8.07e-7	89709.0	89855.0	89854.4
0.4	9999.99	1.79e-3	0.0	9.72e-7	89836.1	89918.3	89918.0
0.5	10000.0	1.69e-3	0.0	6.62e-7	89895.4	89947.9	89947.8
0.6	10000.0	8.72e-6	0.0	6.06e-9	89928.2	89964.8	89964.6
0.7	9999.99	1.47e-5	0.0	9.57e-9	89948.6	89975.8	89975.5
0.8	10000.0	8.92e-3	0.0	1.01e-6	89962.3	89982.9	89982.9
0.9	9999.99	1.28e-2	0.0	9.67e-7	89971.6	89987.9	89987.9
1.0	9999.98	3.45e-2	0.0	1.8e-6	89978.1	89991.6	89991.4
1.1	10000.0	4.36e-2	0.0	1.64e-6	89982.8	89993.9	89993.8
1.2	9999.97	6.6e-2	0.0	1.84e-6	89986.2	89995.5	89995.5
1.3	10000.1	8.67e-2	0.0	1.87e-6	89988.7	89996.8	89996.6
1.4	10000.0	9.6e-2	0.0	1.64e-6	89990.7	89997.4	89997.5
1.5	10000.3	1.58e-1	0.0	2.19e-6	89992.0	89997.8	89997.8
1.6	10000.2	1.7e-1	0.0	1.95e-6	89993.2	89998.2	89998.3
1.7	9999.81	2.59e-1	0.0	2.5e-6	89994.5	89999.6	89999.1
1.8	9999.40	2.91e-1	0.0	2.4e-6	89995.6	89999.9	89999.8
1.9	10000.4	3.73e-1	0.0	2.65e-6	89995.3	89999.2	89998.9
2.0	9999.60	4.83e-1	0.0	2.99e-6	89996.5	90000.1	90000.0
2.1	9998.45	5.83e-1	0.0	3.18e-6	89998.1	90001.3	90001.4
2.2	10001.3	8.51e-1	0.0	4.13e-6	89995.7	89998.4	89998.3
2.3	10000.8	1.01	-1.91e-6	4.38e-6	89996.4	89999.0	89998.9
2.4	10001.0	1.17	0.0	4.56e-6	89996.6	89998.6	89998.8

Table 7: Estimation of envelope parameters (Test 0a)

6.2.4 Test 01

The estimation of the parameters (in the tables, SM) shown in Table 9 for *Curvefit* and Table 10 for *Maximum Likelihood* are good and consistent between them. The AVE values, providing an estimate of the bias, are quite different between the two methods; it seems that estimation of the bias is not an easy task, as it can be expected. The *Fringe Period* results shown in Table 11 highlight the better performances of the densified approach. The error is underestimated, though. For each table, values not satisfing the criteria are in boldface. The criteria parameter $|b|/(\mu \neq 0)$ is always below $1.0 \cdot 10^{-4}$, apart

σ	bg	σ_{bg}	x'_0	$\sigma_{x'_0}$	corrFactA	Integral	Err function
0.2	9999.99	2.17e-3	0.0	2.2e-6	89347.2	89673.3	89673.1
0.3	9999.99	1.24e-3	0.0	9.08e-7	89709.0	89855.1	89854.4
0.4	9999.99	1.29e-3	0.0	7e-7	89836.2	89918.2	89918.1
0.5	10000.0	2.41e-3	0.0	9.42e-7	89896.9	89949.4	89949.4
0.6	10000.0	3.06e-3	0.0	8.17e-7	89938.1	89974.5	89974.5
0.7	10000.0	4.03e-3	0.0	7.05e-7	89938.1	90003.8	90004.6
0.8	10000.0	1.04e-2	0.0	1.18e-6	90019.3	90038.9	90039.9
0.9	10000.0	1.51e-2	0.0	1.14e-6	90061.0	90075.6	90077.3
1.0	10000.0	2.86e-2	0.0	1.49e-6	90100.2	90111.2	90113.4
1.1	9999.97	3.7e-2	0.0	1.39e-6	90135.8	90143.9	90146.8
1.2	9999.97	6.35e-2	0.0	1.78e-6	90167.2	90173.0	90176.5
1.3	10000.0	7.5e-2	0.0	1.62e-6	90194.5	90198.3	90202.4
1.4	9999.92	9.37e-2	0.0	1.6e-6	90218.3	90220.7	90225.2
1.5	9999.89	1.47e-1	0.0	2e-6	90238.9	90240.2	90244.8
1.6	9999.99	2.02e-1	0.0	2.31e-6	90256.5	90256.6	90261.7
1.7	9999.89	2.6e-1	0.0	2.5e-6	90272.0	90271.5	90276.7
1.8	10000.6	3.61e-1	0.0	2.96e-6	90284.7	90283.2	90288.7
1.9	10000.5	4.46e-1	0.0	3.16e-6	90296.5	90294.6	90300.1
2.0	9998.47	4.8e-1	0.0	2.96e-6	90308.6	90306.3	90312.3
2.1	9998.16	5.19e-1	0.0	2.83e-6	90317.0	90314.3	90320.2
2.2	10002.6	9.4e-1	0.0	4.55e-6	90321.7	90318.3	90324.1
2.3	9998.54	9.75e-1	1.91e-6	4.22e-6	90332.6	90329.0	90335.4
2.4	10001.8	1.11	0.0	4.30e-6	90335.8	90331.4	90337.9

Table 8: Estimation of envelope parameters (Test 0b)

for the non-densified algorithms of *Fringe Period*.

6.2.5 Test 1

The estimation SM of the parameters shown in Table 12 for *Curvefit* and Table 13 for *Maximum Likelihood* are good in spite of the noise and are consistent between them. The AVE values are consistent also, apart for a high value for *bg*. NSD seems correctly distributed only in the case of *Maximum Likelihood*, while *Curvefit* provides severely underestimated errors. The *Fringe Period* results shown in Table 14 do not look different from the non-noisy case.

par	SM	SSD	AVE	STD	NAE	NSD	$\|b\|/(\mu \neq 0)$
x'_0	-2.97e-8	2.31e-14	-2.97e-8	2.97e-8	-5.67e3	5.67e3	3.0e-8
y'_0	-8.17e-9	5.33e-15	-8.17e-9	8.17e-9	-2.7e-2	2.7e-2	8.2e-9
σ_x	1.50	1.19e-7	2.38e-7	2.38e-7	2.02e2	2.02e2	1.6e-7
σ_y	1.50	1.90e-6	8.80e-5	8.8e-5	2.47e2	2.47e2	5.9e-5
V	1.00	0.00	1.19e-7	1.19e-7	1.80e2	1.80e2	1.2e-7
ϕ	-3.44e-6	2.27e-12	-3.44e-6	3.44e-6	-5.20e3	5.20e3	3.4e-6
bg	1.0001e4	9.76e-3	7.27e-1	7.27e-1	1.07e2	1.07e2	7.3e-5

Table 9: Test 01 results - IDL curvefit

par	SM	SSD	AVE	STD	NAE	NSD	$\|b\|/(\mu \neq 0)$
x'_0	9.41e-18	0.00	9.41e-18	9.41e-18	1.30e-13	1.30e-13	9.4e-18
y'_0	-8.17e-9	5.33e-15	-8.17e-9	8.17e-9	-2.7e-2	2.7e-2	8.2e-9
σ_x	1.50	2.38e-7	3.58e-7	3.58e-7	3.13e-3	3.13e-3	2.4e-7
σ_y	1.50	0.00	9.76e-5	9.76e-5	7.84e-2	7.84e-2	6.5e-5
V	1.00	0.00	-5.96e-8	5.96e-8	-2.18e-3	2.18e-3	6.0e-8
ϕ	-4.55e-15	2.96e-21	-4.55e-15	4.55e-15	-4.98e-13	4.98e-13	4.6e-15
bg	1.0000e4	1.07e-2	-3.61e-2	3.61e-2	-4.08e-4	4.08e-4	3.6e-6

Table 10: Test 01 results - Maximum Likelihood

6.2.6 Test 2

The estimation SM of the parameters shown in Table 15 for *Curvefit* and Table 16 for *Maximum Likelihood* are again in good agreement. The AVE values are consistent; now both methods show a underestimation of bg. Again NSD seems correctly distributed only in the case of *Maximum Likelihood*, apart for the slightly high value for x'_0, while *Curvefit* still provides severely underestimated errors. The *Fringe Period* results shown in Table 17 are very stable through

par	SM	SSD	AVE	STD	NAE	NSD	$b/(\mu \neq 0)$
FP (1d)	4.9717e-2	1.86e-8	1.66e-5	1.66e-5	5.88e-1	5.88e-1	**3.3e-3**
FP (1d hd)	4.9700e-2	7.45e-9	8.20e-8	8.20e-8	1.29e-2	1.29e-2	1.6e-6
FP (2d)	4.9717e-2	6.33e-8	1.66e-5	1.66e-5	4.55	4.55	**3.3e-3**
FP (2d hd)	4.9700e-2	7.45e-9	7.08e-8	7.08e-8	9.49e-2	9.49e-2	1.4e-6

Table 11: Test 01 results - Fringe Period

par	SM	SSD	AVE	STD	NAE	NSD
x'_0	2.40e-6	7.18e-5	2.40e-6	7.18e-5	**-3.10e4**	**1.22e6**
y'_0	2.92e-7	7.48e-5	2.92e-7	7.48e-5	5.11e-2	**1.31e1**
σ_x	1.50	1.22e-4	2.19e-6	1.22e-4	**-6.17e2**	**1.06e4**
σ_y	1.50	1.26e-3	1.12e-4	1.27e-3	**1.14**	**1.32e1**
V	1.00	3.45e-5	-2.18e-7	3.45e-5	**-1.11e1**	**5.47e3**
ϕ	3.00e-4	9.07e-3	3.01e-4	9.08e-3	**-3.12e4**	**1.22e6**
bg	9.999e3	8.95e1	-1.20	8.95e1	**-1.32e-1**	**1.30e1**

Table 12: Test 1 results - IDL curvefit

par	SM	SSD	AVE	STD	NAE	NSD
x'_0	2.19e-6	7.15e-5	2.19e-6	7.15e-5	3.03e-2	9.88e-1
y'_0	7.25e-7	7.33e-5	7.26e-7	7.33e-5	9.82e-3	9.92e-1
σ_x	1.50	1.17e-4	6.07e-7	1.17e-4	5.09e-3	1.02
σ_y	1.50	1.25e-3	1.04e-4	1.25e-3	8.13e-2	1.00
V	1.00	2.68e-5	-4.46e-7	2.68e-5	-1.6e-2	9.81e-1
ϕ	2.75e-4	9.04e-3	2.75e-4	9.04e-3	3.00e-2	9.88e-1
bg	1.0000e4	8.90e1	-6.64e-1	8.89e1	-4.26e-3	1.00

Table 13: Test 1 results - Maximum Likelihood

the tests. Now the NSD value is closer to the correct distribution, but still one order of magnitude off the expected value of 1. There is no evidence of a significant difference with respect to test 1 results.

6.2.7 Test 3

There is no evidence of a significant difference with respect to the previous results, meaning that the variability of the parameters among the images is

par	SM	SSD	AVE	STD	NAE	NSD
FP (1d)	4.9717e-2	2.75e-7	1.66e-5	1.66e-5	**5.88e-1**	**5.88e-1**
FP (1d hd)	4.9700e-2	4.33e-7	8.10e-8	8.30-8	3.35e-2	**8.85e-2**
FP (2d)	4.9717e-2	2.76e-7	1.66e-5	1.66e-5	**4.55**	**4.55**
FP (2d hd)	4.9701e-2	4.41e-7	7.31e-8	7.58e-8	9.52e-2	**9.86e-2**

Table 14: Test 1 results - Fringe Period

par	SM	SSD	AVE	STD	NAE	NSD
x'_0	1.01e-6	8.30e-5	3.06e-7	1.04e-4	**6.86e4**	**1.47e6**
y'_0	1.44e-2	7.50e-5	1.05e-7	7.50e-5	1.81e-2	**1.31e1**
σ_x	1.50	1.24e-4	2.10e-6	1.24e-4	**4.31e1**	**7.10e3**
σ_y	1.50	1.26e-3	1.20e-4	1.26e-3	**1.23**	**1.33e1**
V	8.10e-1	3.75e-5	-3.35e-7	3.74e-5	**2.39e2**	**5.78e3**
ϕ	7.81e-3	9.39e-3	2.11e-4	9.39e-3	**-2.70e4**	**1.14e6**
bg	9.999e3	8.93e1	-1.80	8.93e1	**-2.20e-1**	**1.30e1**

Table 15: Test 2 results - IDL curvefit

	SM	SSD	AVE	STD	NAE	NSD
x'_0	1.32e-6	8.14e-5	6.16e-7	1.03e-4	8.43e-3	**1.41**
y'_0	1.44e-2	7.35e-5	5.33e-7	7.35e-5	7.18e-3	9.93e-1
σ_x	1.50	1.19e-4	5.29e-7	1.19e-4	4.35e-3	1.03
σ_y	1.50	1.24e-3	1.13e-4	1.25e-3	8.89e-2	1.01
V	8.10e-1	3.36e-5	-5.11e-7	3.36e-5	-1.49e-2	9.83e-1
ϕ	7.85e-3	9.13e-3	2.50e-4	9.13e-3	2.71e-2	9.88e-1
bg	9.999e3	8.88e1	-1.31	8.8e1	-1.16e-2	1.00

Table 16: Test 2 results - Maximum Likelihood

either too small to be significant or not affecting the results.

6.2.8 Test 4

Again no apparent significant change with respect to the previous tests.

par	SM	SSD	AVE	STD	NAE	NSD
FP (1d)	4.9717e-2	3.37e-7	1.68e-5	1.67e-5	**5.94e-1**	**5.94e-1**
FP (1d hd)	4.9701e-2	2.26e-7	-1.09e-7	1.12e-7	-5.44e-2	**1.42e-1**
FP (2d)	4.9717e-2	3.36e-7	1.68e-5	1.68e-5	**4.60e0**	**4.60e0**
FP (2d hd)	4.9701e-2	2.36e-7	-1.20e-7	1.22e-7	**-1.56e-1**	**1.59e-1**

Table 17: Test 2 results - Fringe Period

par	SM	SSD	AVE	STD	NAE	NSD
x'_0	3.23e-5	7.62e-5	6.16e-5	9.79e-5	**1.96e5**	**1.80e6**
y'_0	-6.36e-4	1.52e-2	2.69e-7	7.51e-5	4.67e-2	**1.31e1**
σ_x	1.50	1.25e-4	1.88e-6	1.23e-4	**1.32e2**	**8.43e3**
σ_y	1.50	1.26e-3	1.18e-4	1.26e-3	**1.21**	**1.32e1**
V	8.52e-1	2.87e-2	-3.76e-7	3.67e-5	**-9.02e1**	**5.26e3**
ϕ	5.86e-4	1.03e-1	3.73e-4	9.63e-3	**-3.09e3**	**1.42e6**
bg	9.998e3	8.92e1	-1.66	8.92e1	**-1.99e-1**	**1.30e1**

TABLE 18: Test 3 results - IDL curvefit

par	SM	SSD	AVE	STD	NAE	NSD
x'_0	3.15e-5	7.20e-5	6.08e-5	9.42e-5	**8.32e-1**	**1.29**
y'_0	-6.36e-4	1.52e-2	7.49e-7	7.35e-5	1.01e-2	9.93e-1
σ_x	1.50	1.20e-4	4.37e-7	1.19e-4	3.62e-3	1.03
σ_y	1.50	1.25e-3	1.11e-4	1.25e-3	8.67e-2	1.01
V	8.52e-1	2.87e-2	-3.90e-7	3.25e-5	-1.29e-2	9.85e-1
ϕ	4.87e-4	1.03e-1	2.74e-4	9.11e-3	2.96e-2	9.87e-1
bg	9.999e3	8.87e1	-1.12	8.87e1	-9.48e-3	1.00

TABLE 19: Test 3 results - Maximum Likelihood

par	SM	SSD	AVE	STD	NAE	NSD
FP (1d)	4.9718e-2	8.78e-6	1.67e-5	2.02e-5	**5.94e-1**	**7.19e-1**
FP (1d hd)	4.9701e-2	6.82e-6	-1.14e-7	9.87e-6	**-8.46e-1**	**9.46**
FP (2d)	4.9719e-2	8.78e-6	1.67e-5	2.02e-5	**4.60**	**5.57**
FP (2d hd)	4.9701e-2	6.82e-6	-1.23e-7	9.88e-6	**-1.49e-1**	**1.29e1**

TABLE 20: Test 3 results - Fringe Period

par	SM	SSD	AVE	STD	NAE	NSD
x'_0	2.22e-6	8.50e-5	1.51e-6	1.06e-4	**8.54e4**	**3.95e6**
y'_0	-6.36e-4	1.52e-2	2.72e-7	7.51e-5	4.74e-2	**1.31e1**
σ_x	1.50	1.25e-4	1.89e-6	1.24e-4	**4.62e2**	**1.55e4**
σ_y	1.50	1.26e-3	1.18e-4	1.26e-3	**1.21**	**1.32e1**
V	8.52e-1	2.87e-2	-3.75e-7	3.68e-5	**-9.83e1**	**9.11e3**
ϕ	5.79e-4	1.03e-1	3.6e-4	9.62e-3	**4.31e4**	**2.03e6**
bg	9.998e3	8.92e1	-1.65	8.92e1	**-1.98e-1**	**1.30**

TABLE 21: Test 4 results - IDL curvefit

	SM	SSD	AVE	STD	NAE	NSD
x'_0	1.47e-6	8.13e-5	7.73e-7	1.03e-4	1.05e-2	**1.41**
y'_0	-6.36e-4	1.52e-2	7.52e-7	7.35e-5	1.01e-2	9.93e-1
σ_x	1.50	1.20e-4	4.47e-7	1.19e-4	3.70e-3	1.03
σ_y	1.50	1.25e-3	1.11e-4	1.25e-3	8.66e-2	1.01
V	8.52e-1	2.87e-2	-3.96e-7	3.25e-5	-1.31e-2	9.85e-1
ϕ	4.86e-4	1.03e-1	2.73e-4	9.11e-3	2.95e-2	9.87e-1
bg	9.999e3	8.87e1	-1.12	8.87e1	-9.41e-3	1.00

TABLE 22: Test 4 results - Maximum Likelihood

par	SM	SSD	AVE	STD	NAE	NSD
FP (1d)	4.9718e-2	8.78e-6	1.67e-5	2.02e-5	**5.94e-1**	**7.19e-1**
FP (1d hd)	4.9701e-2	6.82e-6	-1.14e-7	9.88e-6	**-3.52e-1**	**1.09e1**
FP (2d)	4.9718e-2	8.78e-6	1.67e-5	2.02e-5	**4.60**	**5.56**
FP (2d hd)	4.9701e-2	6.82e-6	-1.23e-7	9.88e-6	**-1.44e-1**	**1.29e1**

TABLE 23: Test 4 results - Fringe Period

7 Conclusions

From the analysis of the test results, we can see that both the *Maximum Likelihood* and the *Curvefit* converge quickly to a local minimum. In presence of noise on the image (photon noise) and on the parameters, the standard errors are in some cases underestimated, as highlighted by the chosen quality estimators. This is particularly true for *Curvefit* and *Fringe Period*. This can be corrected only by choosing better weighting functions. However, the *Maximum Likelihood* results are in good agreement with the expected performance and statistical properties.